A FUNDAMENTAL PARAMETRIC REPRESENTATION OF SPACE CURVES.*

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Introduction.—In 1848 J. A. Serret † proposed the problem of solving the differential equation

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{1}$$

where s, x, y, z are functions of a single parameter, and he developed a general method of solution, without giving, however, in simple explicit form the general solution of this equation. The same problem has been considered by Darboux ‡ on two occasions; his solution makes use of the idea of curves with parallel tangents. Applying this method he gets rather complicated forms for the solution of (1). The author, while studying certain surfaces whose middle surfaces are surfaces of translation, was brought incidentally to the following form of solution

$$\begin{aligned} x &= \varphi - u\varphi' + \psi', \\ iy &= \varphi - u\varphi' - \psi', \\ z &= \varphi' + u\psi' - \psi, \end{aligned} \tag{2}$$

and

$$s = \varphi' - u\psi' + \psi, \tag{3}$$

where φ and ψ are functions of u and the primes indicate differentiation with respect to u. However, five years ago Montcheuil § was brought to a similar result in the same incidental manner. Salkowski ‖ made use of this result of Montcheuil in the discussion of certain problems. The present paper deals with an exposition of this form of defining a space

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*Presented to the American Mathematical Society, April 28, 1911.

†Sur l'integration de l'equation $dx^2 + dy^2 + dz^2 = ds^2$, Journal de Mathématiques, vol. 13 (1848), pp. 353–360.

‡Sur l'integration de l'equation $dx^2 + dy^2 = ds^2$ et de quelques équations analogues, Journal de Mathématiques, ser. 2, vol. 18 (1873), pp. 236–240; also Sur la resolution de l'equation $dx^2 + dy^2 + dz^2 = ds^2$ et de quelques équations analogues, Journal de Mathématiques, ser. 4, vol. 3 (1887), pp. 305–325.

§Résolution de l'equation $ds^2 = dx^2 + dy^2 + dz^2$, Bulletin de la Société Mathématique de France, vol. 33 (1905), pp. 170–171.

‖Ueber algebraisch rektifizierbare Raumkurven, Mathematische Annalen, vol. 67 (1909), pp. 445–458.

curve and with an investigation of certain problems for which this parametric form is very suitable. Before proceeding to an indication of the various topics discussed, it should be remarked that equations (2) and (3) give in explicit form the coördinates of minimal curves in four-space, which constitute an interesting generalization of the Weierstrass * formulas for minimal curves in three-space.

In §1 we show that it is possible to put the equations of any curve, *not a straight line*, in the form (2), and in general in two ways. We shall refer to (2) as the *normal form* of the equations, u the *normal parameter*, and φ and ψ the corresponding *normal functions*. Other normal parameters and functions of a similar type are discussed in §2. The expressions for the curvature, torsion and the direction-cosines of the tangent, principal normal and binormal are found in §3, together with a discussion of curves of zero curvature.

In §4 the general problem of the congruence of non-minimal curves is stated in terms of normal functions and parameters, and in §7 this problem is reduced to the integration of Schwarzian equations. The results are similar to those obtained recently by Study,† from a somewhat different point of view. In §8 the methods are applied to the study of the exceptional case of minimal curves. Curves in a real plane are considered in §5, and curves on the minimal cone in §6.

The question of algebraic curves is considered in §9, as well as the determination of a transformed form of equations (2) which is suitable for the discussion of real curves.

1. **Determination of u, φ, ψ.**—From (2) and (3) we have

$$\frac{dx + idy}{dz + ds} = -\frac{dz - ds}{dx - idy} = -u. \tag{4}$$

Suppose now that we have a curve defined by

$$x = f_1(v), \quad y = f_2(v), \quad z = f_3(v), \tag{5}$$

where f_1, f_2, f_3 are analytic functions of v for a certain domain of the latter. For the present we assume that

$$f_1 - if_2 \neq \text{const.} \tag{6}$$

If we put

$$t = (f_1'^2 + f_2'^2 + f_3'^2)^{\frac{1}{2}}, \tag{7}$$

*Monatsberichte der Berliner Akademie (1866), p. 619; also Eisenhart, Differential Geometry, p. 49 (hereafter a reference to this book will be in the form E. p. 49).

†Zur Differentialgeometrie der analytischen Curven, Transactions of the American Mathematical Society, vol. 10 (1909), pp. 1–49; also Die Natürlichen Gleichungen der analytischen Curven in Euklidischen Raume, Transactions of the American Mathematical Society, vol. 11 (1910), pp. 249–279.

where the primes denote differentiation with respect to v, then in comparison with (4) we note that the functions u, φ, ψ defined by

$$(8) \qquad \frac{f_3' - t}{f_1' - if_2'} = u, \quad \tfrac{1}{2}(f_1 + if_2) = \varphi - u\frac{d\varphi}{du}, \quad \tfrac{1}{2}(f_1 - if_2) = \frac{d\psi}{du}.$$

enable us to write equations (5) in the form (2),* and thus are a normal parameter and normal functions.

Furthermore, the equations

$$(9) \qquad \frac{f_3' + t}{f_1' - if_2'} = \bar{u}, \quad \tfrac{1}{2}(f_1 + if_2) = \bar{\varphi} - \bar{u}\frac{d\bar{\varphi}}{d\bar{u}}, \quad \tfrac{1}{2}(f_1 - if_2) = \frac{d\bar{\psi}}{d\bar{u}},$$

determine a second normal parameter $\bar{u}$ and normal functions $\bar{\varphi}$ and $\bar{\psi}$ by means of which the equations of the curve can be given the normal form. It is evident from their manner of definition that there are only two sets of normal parameters and functions.

From (8) and (9) it follows that $u = \bar{u}$, when $t = 0$, that is when C is minimal. In this case,

$$(10) \qquad \varphi'' - u\psi'' = 0.$$

We proceed to the determination of the relations between the two sets u, φ, ψ and $\bar{u}$, $\bar{\varphi}$, $\bar{\psi}$. Equations (8) are identically satisfied by

$$(11) \qquad \begin{aligned} f_1 &= \varphi - u\varphi' + \psi', \quad if_2 = \varphi - u\varphi' - \psi', \\ f_3 &= \varphi' + u\psi' - \psi, \quad t = u\psi'' - \varphi'', \end{aligned}$$

and equations (9) reduce to

$$(12) \qquad \bar{u} = \frac{\varphi''}{\psi''}, \quad \bar{\varphi} - \bar{u}\bar{\varphi}' = \varphi - u\varphi', \quad \bar{\psi}' = \psi',$$

where the primes refer to differentiation with respect to the corresponding normal parameter. If the last two equations be differentiated with respect to u and the resulting equations be divided, the result is reducible by (12) to

$$(13) \qquad u = \frac{\bar{\varphi}''}{\bar{\psi}''},$$

which is in keeping with the symmetry of the equations.

From the last of (12) we have, by means of the first of (12),

$$(14) \qquad \bar{\psi} = \int \psi' d\bar{u} = \int \psi' \left(\frac{\varphi''}{\psi''}\right)' du = \psi'\left(\frac{\varphi''}{\psi''}\right) - \varphi',$$

* It should be observed that ϕ is determined thus only to within the additive function cu, where c is an arbitrary constant; the effect of taking $c \neq 0$ is merely to increase z by c. A similar effect is produced by the undetermined additive constant in ψ.

where we have disregarded an additive constant. In similar manner we have from (13)

$$\bar{\varphi} = \varphi - u\varphi' + (u\,\psi' - \psi)\frac{\varphi''}{\psi''}. \tag{15}$$

Furthermore from these values we find

$$\bar{s} = \bar{\varphi}' - \bar{u}\bar{\psi}' + \bar{\psi} = -s. \tag{16}$$

It remains for us to consider the exceptional case (6) which thus far has been excluded from the discussion. In this case we replace equations (8) and (9) by the unique set

$$\frac{f_1' + if_2'}{2f_3'} = -u, \quad \psi = cu + d, \quad \frac{d\varphi}{du} = f_3 + d, \tag{17}$$

where c and d denote constants.

Unless $t = 0$, equations (8) and (9) may be written

$$\frac{\gamma - 1}{\alpha - i\beta} = u, \quad \frac{\gamma + 1}{\alpha - i\beta} = \bar{u}, \tag{18}$$

where α, β, γ denote the direction-cosines of the tangent.

From (18) it follows that in the case of a non-minimal straight line, that is a curve whose tangent has the same direction at every point and for which $ds \neq 0$, both u and $\bar{u}$ are constant, and conversely. The same is true, furthermore, when the straight line is minimal, as follows from the general equations of such a line, namely *

$$x = \frac{1 - a^2}{2}v, \quad y = i\frac{1 + a^2}{2}v, \quad z = av, \tag{19}$$

with the difference that in this case u and $\bar{u}$ are equal to one another. Hence it is impossible to give a straight line parametric representation of the form (1).

We proceed to the consideration of the case where one of the normal parameters is constant, say $\bar{u} = c$ Then from (12) we have

$$\varphi'' = c\,\psi''. \tag{20}$$

In this case equations (2) reduce to

$$\begin{aligned} x &= c\psi - \psi'(cu - 1) + b, \\ iy &= c\psi - \psi'(cu + 1) + b, \\ z &= (c + u)\psi' - \psi + a, \end{aligned} \tag{21}$$

where a and b are constants. It is readily shown that the curve defined by (21) lies in the isotropic plane

*E. p. 48.

$$(1 - c^2)x + i(1 + c^2)y + 2cz = A, \tag{22}$$

where A is a determinate constant. Conversely, equation (22) defines the general isotropic plane, and the condition that the functions (2) satisfy this equation is reducible to

$$(u - c)(\varphi'' - c\psi'') = 0. \tag{23}$$

Hence (20) is the necessary and sufficient condition that the curve lie in an isotropic plane. It should be remarked that (17) corresponds to a special case of (20).

Gathering together these results we have the theorem:

When an analytic curve is defined by equations

$$x = f_1(v), \quad y = f_2(v), \quad z = f_3(v),$$

for values of v within a domain R, the functions u, φ, ψ determined by (8) *lead to a representation of the curve in the normal form. In general there is a second set of normalizing functions defined by*

$$\bar{u} = \frac{\varphi''}{\psi''}, \quad \bar{\varphi} = \varphi - u\varphi' + (u\psi' - \psi)\frac{\varphi''}{\psi''}, \quad \bar{\psi} = \frac{\psi'\varphi''}{\psi''} - \varphi'. \tag{24}$$

When the curve is minimal, the two sets of functions are the same and conversely. When the curve lies in an isotropic plane, one of the normal parameters is constant, and conversely. For a straight line, minimal or otherwise, the representation is impossible.

In illustration of the foregoing we observe that for the circular helix

$$x = a \cos v, \quad y = a \sin v, \quad z = bv,$$

the normal parameters and functions are of the form

$$u = \frac{ic}{a}e^{iv}, \quad c = b - \sqrt{a^2 + b^2},$$

$$\varphi(u) = \frac{1}{2}\frac{a^2u}{ic}\left(1 - \log\frac{iau}{c}\right),$$

$$\psi(u) = \frac{i}{2}c\left(1 + \log\frac{iau}{c}\right).$$

The parameter $\bar{u}$ is obtained by replacing c by $\bar{c}$, where

$$\bar{c} = b + \sqrt{a^2 + b^2}.$$

The functions $\bar{\varphi}$ and $\bar{\psi}$ are of the same form as the above with $\bar{c}$ and $\bar{u}$ in place of c and u respectively.

2. **Other Normal Parameters.**—We have observed that there are at most two normal parameters in terms of which the equations of a curve may be given the form (2). In stating this fact we mean the precise form (2), for, as we shall show, there are other normal forms analogous to (2). In fact, we shall prove that it is possible to express the coordinates in a similar form in terms of a parameter u_1, defined by

$$u_1 = \frac{au + b}{cu + d},$$

where a, b, c, d are constants, which without loss of generality may be chosen to satisfy the condition

$$ad - bc = 1.$$

To this end we observe that the equations

$$(au + b)\varphi'' du = u_1 \varphi_1''(u_1) du_1,$$

$$(cu + d)\varphi'' du = \varphi_1''(u_1) du_1,$$

and like ones in ψ and ψ_1, are consistent and may be replaced by

$$u_1 = \frac{au + b}{cu + d}, \quad \varphi_1(u_1) = \frac{\varphi}{cu + d}, \quad \psi_1(u_1) = \frac{\psi}{cu + d}.$$

Solving these equations for u, φ and ψ, we obtain

$$u = \frac{b - du_1}{cu_1 - a}, \quad \varphi = \frac{\varphi_1}{a - cu_1}, \quad \psi = \frac{\psi_1}{a - cu_1}.$$

When these values are substituted in (2) and (3), the latter become

$$\begin{aligned} x &= d(\varphi_1 - u_1\varphi_1') + b\varphi_1' + a\psi_1' + c(\psi_1 - u_1\psi_1'), \\ iy &= d(\varphi_1 - u_1\varphi_1') + b\varphi_1' - a\psi_1' - c(\psi_1 - u_1\psi_1'), \\ z &= a\varphi_1' + c(\varphi_1 - u_1\varphi_1') - b\psi_1' - d(\psi_1 - u_1\psi_1'), \\ s &= a\varphi_1' + c(\varphi_1 - u_1\varphi_1') + b\psi_1' + d(\psi_1 - u_1\psi_1'), \end{aligned}$$

which expressions are a generalization of (2) and (3).

In order to interpret this result, we observe that if s be replaced by it in (3) and in the foregoing equations, these equations and (2) define in terms of the respective parameters u and u_1 a minimal curve in Euclidean four-space, as follows from (1).

If we denote by x_1, y_1, z_1, t_1 the coördinates of a minimal curve for

which u_1 is the normal parameter in the sense of §1 and φ_1, ψ_1 are the normal functions, so that

$$x_1 = \varphi_1 - u_1\varphi_1' + \psi_1', \quad iy_1 = \varphi_1 - u_1\varphi_1' - \psi_1',$$
$$z_1 = \varphi_1' + u_1\psi_1' - \psi_1, \quad it_1 = \varphi_1' - u_1\psi_1' + \psi_1,$$

the preceding set of equations is expressible in the form

$$\begin{aligned} x &= \tfrac{1}{2}[(a + d)x_1 + (d - a)iy_1 + (b - c)z_1 + (b + c)it_1], \\ y &= \tfrac{1}{2}[(a - d)ix_1 + (a + d)y_1 - (b + c)iz_1 + (b - c)t_1], \\ z &= \tfrac{1}{2}[(c - b)x_1 + (b + c)iy_1 + (a + d)\, z_1 + (a - d)it_1], \\ t &= \tfrac{1}{2}[-(b + c)ix_1 + (c - b)y_1 + (d - a)iz_1 + (a + d)t_1]. \end{aligned}$$

Since these equations define an orthogonal transformation, of determinant $+ 1$, of four-space, we have the theorem:

When the equations of a curve of Euclidean three-space are expressed in the form (2) in terms of a normal parameter u, it is possible without quadratures to express the coördinates in analogous form in terms of a parameter u_1, *where*

$$u_1 = \frac{au + b}{cu + d},$$

a, b, c, d, being constants satisfying the condition $ad - bc = 1$. *This transformation of parameters may be effected by the rotation of the minimal curve of coördinates* $(x, y, z, - is)$ *in Euclidean four-space.**

When the curve is defined in terms of a general parameter v as in §1, the parameter u_1 is given by

$$u_1 = \frac{a(f_3' - t) + b(f_1' - if_2')}{c(f_3' - t) + d(f_1' - if_2')}.$$

Moreover, the second parameter $\bar{u}^1$ is given by

$$\bar{u}_1 = \frac{a\bar{u} + b}{c\bar{u} + d} = \frac{a\varphi'' + b\psi''}{c\varphi'' + d\psi''} = \frac{a\varphi_1'' + b\psi_1''}{c\varphi_1'' + d\psi_1''}.$$

3. Radii of Curvature and Torsion. Curves of Zero Curvature.—The equation

$$(25) \qquad \frac{1}{\rho^2} = \frac{\left(\frac{d^2x}{du^2}\right)^2 + \left(\frac{d^2y}{du^2}\right)^2 + \left(\frac{d^2z}{du^2}\right)^2 - \left(\frac{d^2s}{du^2}\right)^2}{\left(\frac{ds}{du}\right)^4},$$

*The converse of this result is true, as will be shown in a subsequent paper dealing with minimal curves and surfaces in Euclidean four-space.

giving the radius of curvature of a curve, is independent of the real or complex character of the parameter u.* When the expressions from (2) and (3) are substituted in (25), it becomes

$$\frac{1}{\rho^2}=\frac{4(\varphi'''\psi''-\varphi''\psi''')}{(\varphi''-u\psi'')^4}. \tag{26}$$

It is equally true that the derivation of the Frenet formulas †

$$\frac{d\alpha}{ds}=\frac{l}{\rho},\quad \frac{dl}{ds}=-\left(\frac{\alpha}{\rho}+\frac{\lambda}{\tau}\right),\quad \frac{d\lambda}{ds}=\frac{l}{\tau} \tag{27}$$

is not conditioned by the character of the parameter u. In (27) α, l, λ denote the direction-cosines of the tangent, principal-normal and binormal with respect to the x-axis; with respect to the other two axes they are β, m, μ and γ, n, ν, respectively, and these direction-cosines also satisfy (27). Furthermore τ denotes the radius of torsion. Its expression, derived from‡

$$\frac{1}{\tau}=-\frac{\rho^2}{\left(\frac{ds}{du}\right)^6}\begin{vmatrix}\frac{dx}{du} & \frac{dy}{du} & \frac{dz}{du}\\ \frac{d^2x}{du^2} & \frac{d^2y}{du^2} & \frac{d^2z}{du^2}\\ \frac{d^3x}{du^3} & \frac{d^3y}{du^3} & \frac{d^3z}{du^3}\end{vmatrix}, \tag{28}$$

is reducible in terms of the normal functions and parameter to

$$\frac{1}{\tau}=i\left[\frac{d\log\rho}{ds}+\frac{(u\psi''-\varphi'')'}{(u\psi''-\varphi'')^2}+\frac{2\psi''}{(u\psi''-\varphi'')^2}\right], \tag{29}$$

where the primes indicate differentiation with respect to u.

From (2) and (3) we find the following expressions for the direction-cosines of the tangent,

$$\alpha=(\psi''-u\varphi'')A,\quad \beta=i(\psi''+u\varphi'')A,\quad \gamma=(\varphi''+u\psi'')A, \tag{30}$$

where for the sake of brevity we have put

$$A=(\varphi''-u\psi'')^{-1}. \tag{31}$$

By means of (27) we obtain also

$$\begin{gathered} l=\rho A^3[B(1-u^2)-(\varphi''^2-\psi''^2)],\quad m=i\rho A^3[B(1+u^2)+\varphi''^2+\psi''^2],\\ n=2\rho A^3[Bu+\varphi''\psi''], \end{gathered} \tag{32}$$

*Cf. E. pp. 9, 10.
†E. p. 17.
‡E. p. 21, Ex. 11.

and

$$\lambda = -i\rho A^3[B(1-u^2)+(\varphi''^2-\psi''^2)], \quad \mu = \rho A^3[B(1+u^2)-(\varphi''^2+\psi''^2)], \tag{33}$$
$$\nu = -2i\rho A^3(Bu - \varphi''\psi''],$$

where we have put

$$B = \varphi''\psi''' - \varphi'''\psi''. \tag{34}$$

From (26) it follows that the necessary and sufficient condition that the first curvature of a curve be zero is that (20) be satisfied, or that either φ'' or ψ'' be zero. We have seen that when (20) is satisfied the curve lies in the plane (22). Furthermore, when φ'' or ψ'' is zero the curve lies in the plane $x + iy =$ const. or $x - iy =$ const. respectively. The converse of these results being true we have the theorem*

The necessary and sufficient condition that the first curvature of a curve be zero at all its points is that it be a straight line or a curve in an isotropic plane.

4. **Congruence of Space Curves.**—By analytical processes, which are independent of the character of the parameter in terms of which a given curve is defined, it can be shown that a necessary and sufficient condition that two non-minimal space curves be congruent, in the Euclidean sense, is that one of the following three sets of equations be satisfied by the linear element and the radii of curvature and torsion of the two curves:

$$\frac{d\rho}{ds} = f(\rho), \quad \tau = \varphi(\rho); \quad \rho = \text{const.}, \quad \frac{d\tau}{ds} = \varphi(\tau); \quad \rho = \text{const.}, \quad \tau = \text{const.},$$

the functions f and φ, or the constants as the case may be, being the same for both curves.†

If we exclude the case where either ρ or τ is constant, or both, it follows that if two curves are defined in terms of parameters v_1 and v_2 respectively the necessary and sufficient condition that the two curves be congruent, in the Euclidian sense, is that the equations

$$\rho_1(v_1) = \rho_2(v_2), \quad \tau_1(v_1) = \tau_2(v_2), \quad s_1(v_1) = s_2(v_2), \quad \frac{d\rho_1}{ds_1} = \frac{d\rho_2}{ds_2}$$

be consistent. Evidently one of the last two is a consequence of the other and of the first two. Hence, ρ, τ, s and $d\rho/ds$ constitute a set of *characteristic invariants*, to use the terminology of Study.‡

From (16) it is seen that the interchange of the normal parameters u and $\bar{u}$ effects a change of sign in the linear element of the curve. Hence

*Cf. E. Chapter 1.

†Lie, Vorlesungen uber Continuierliche Gruppen, Leipzig, 1893, pp. 684–686; also, Scheffers, Anwendung der Differential und Integral Rechnung auf Geometrie, Leipzig, 1902, vol. 1, p. 207.

‡L. c., p. 24.

the question of sign does not in general cause any difficulty in determining the curves congruent to a given curve, when the latter is defined by equations in the normal form. In consequence of (3), (26) and (29) we have accordingly the theorem:

The problem of finding curves congruent to a non-minimal space curve C, with its equations in the normal form, reduces to the solution of the equations obtained by equating the functions

$$\frac{\varphi'''\psi'' - \varphi''\psi'''}{(\varphi'' - u\psi'')^4}, \quad \frac{(u\psi'' - \varphi'')' + 2\psi''}{(u\psi'' - \varphi'')^2},$$
$$(35) \qquad \varphi' - u\psi' + \psi, \quad \left(\frac{\varphi'''\psi'' - \varphi''\psi'''}{(\varphi'' - u\psi'')^4}\right)'(\varphi'' - u\psi'')^{-1},$$

to similar expressions in u_1, φ_1, ψ_1, where the latter are to be determined; of these equations the first two and either one of the last two constitute a sufficient set. Incidentally we remark that the second of (35) is the function $i/\tau + 1/\rho \, d\rho/ds$, to within algebraic sign. Hence from the present point of view this invariant takes the place of τ which is usually used. A fuller significance of this will be seen later (§7).

5. **Curves in a Real Plane.**—If we put

$$\varphi' + u\psi' - \psi = 0,$$

the curve (2) lies in the plane $z = 0$. In this case equations (2) reduce to

$$x = 2\int \psi \, du - 2u\psi + (1 + u^2)\psi',$$
$$iy = 2\int \psi \, du - 2u\psi - (1 - u^2)\psi',$$

and the arc is given by

$$s = 2(\psi - u\psi').*$$

Furthermore, the curvature is given by

$$\rho^2 = -4u^4\psi''^2,$$

and the arc of the circular indicatrix of the tangent, namely $\sigma = \int \rho^{-1} ds$, is such that

$$u = e^{i\sigma}.$$

From the first equation of this section we find by differentiation that

$$\bar{u} = -u.$$

If the additive constant arising from the integration of the last of equations (12) be taken equal to zero, we have

$$\bar{\psi} = -\psi, \quad \bar{\psi} - \bar{u}\bar{\psi}' = -(\psi - u\psi'), \quad \bar{u}^2\bar{\psi}'' = -u^2\psi'',$$

*When the reader compares the expressions for x, y, z, with (67), he will observe that plane curves may be looked upon readily as minimal curves in three-space.

where in each case the primes indicate differentiation with respect to the argument.

From the foregoing results it follows that the determination of the curves congruent to a plane curve whose equations have the above form consists of the solution of the equations

$$\psi_1 - u_1\psi_1' = \psi - u\psi', \quad u_1^2\psi_1'' = u^2\psi'',$$

and of

$$\psi_1 - u_1\psi_1' = \psi - u\psi', \quad u_1^2\psi_1'' = -u^2\psi'',$$

where ψ_1 is a function of u_1.

The general solution of the former is

$$u_1 = cu, \quad \psi_1 = \psi + eu,$$

and of the latter

$$u_1 = \frac{c}{u}, \quad \psi_1 = \frac{2}{u}\int \psi\, du - \psi + \frac{e}{u},$$

where c and e are arbitrary constants.

When these values are substituted in equations for x_1 and y_1 analogous to those for x and y, the resulting equations are expressible in the forms

$$x_1 = \frac{1}{2c}[(1 + c^2)x - (1 - c^2)iy + 2e],$$

$$y_1 = \frac{1}{2c}[(1 - c^2)ix + (1 + c^2)y + 2ei ,$$

and

$$x_1 = \frac{1}{2c}[(1 + c^2)x + (1 - c^2)iy + 2e],$$

$$y_1 = \frac{1}{2c}[(1 - c^2)ix - (1 + c^2)y + 2ei],$$

respectively.

We shall close this section with the determination of plane curves of constant curvature $1/a$. From the expression for ρ it follows that such curves are characterized by the equation

$$4u^4\psi''^2 = -a^2,$$

from which we obtain

$$\psi = \frac{ai}{2}\log u + bu + c,$$

where b and c are constants.

Substituting in (2) we have

$$x = b + e + \frac{ai}{2u}(1 - u^2), \quad iy = e - b - \frac{ai}{2u}(1 + u^2),$$

so that

$$[x - (b + e)]^2 + [y - i(b - e)]^2 = a^2.$$

§6. **Curves on the Minimal Cone, or Sphere of Zero Radius.** Before considering further the problem of congruence of curves in general it is necessary to study in particular the curves which lie on the minimal cone, or sphere of zero radius, whose equation is

$$x^2 + y^2 + z^2 = 0. \tag{36}$$

The necessary and sufficient condition that a curve whose equations are in the normal form (2) lie on the cone (36) is

$$(\varphi' - u\psi' - \psi)^2 + 4\psi'(\varphi - u\psi) = 0. \tag{37}$$

If we put

$$f^2 = u\psi - \varphi, \tag{38}$$

the preceding equation reduces to

$$f^2(\psi' - f'^2) = 0. \tag{39}$$

When $f = 0$, and thus $\varphi = u\,\psi$, equations (2) become

$$x = (1 - u^2)\psi', \quad y = i(1 + u^2)\psi', \quad z = 2u\psi'. \tag{40}$$

Furthermore, from (26) we have

$$\frac{1}{\rho^2} = 4\,\frac{3\psi''^2 - 2\psi'\psi'''}{(2\psi')^4}, \tag{41}$$

and from (3)

$$s = 2\psi. \tag{42}$$

When $f \neq 0$ in (38), the corresponding equations (2) are reducible by means of (39) to

$$\begin{aligned} x &= -(uf' - f)^2 + f'^2, \\ iy &= -(uf' - f)^2 - f'^2, \\ z &= 2f'(uf' - f). \end{aligned} \tag{43}$$

When one applies (24) to these equations and determines the functions $\bar{u}$, $\bar{\varphi}$, and $\bar{\psi}$, he obtains

$$\bar{u} = u - \frac{f}{f'}, \quad \bar{\varphi} = \frac{1}{f'}(ff' - \psi)(uf' - f), \quad \bar{\psi} = ff' - \psi.$$

From these follows the relation $\bar{\varphi} - \bar{u}\bar{\psi} = 0$, and consequently the expressions (43) are transformed into the forms (40). Conversely, it is readily shown that in terms of the second set of parameters and functions the equations (40) assume the form (43). Hence we have the theorem:

The curves on the cone $x^2 + y^2 + z^2 = 0$ are characterized by equations of the form (40) and there is only one such representation of the curve for a given set of coördinate axes.

We pass to the consideration of the congruence of two curves defined by equations of the form (40). In this case the expressions (35) reduce respectively to

$$(44)\qquad \frac{3\psi''^2 - 2\psi'\psi'''}{(2\psi')^4}, \quad 0, \quad 2\psi, \quad \left(\frac{3\psi''^2 - 2\psi'\psi'''}{(2\psi')^4}\right)' \frac{1}{2\psi'}.$$

If equation (42) be differentiated with respect to s, we obtain

$$(45)\qquad 2\frac{d\psi}{du}\frac{du}{ds} = 1,$$

in consequence of which we find that

$$\frac{3\psi''^2 - 2\psi'\psi'''}{(2\psi')^4} = \frac{2u'u''' - 3u''^2}{4u'^2},$$

where the primes denote differentiation with respect to u and s respectively in the left-hand and right-hand members of the equation. Consequently in this case the general theorem of §4 assumes the following form similar to a theorem due to Study:*

To each Schwarzian equation

$$(46)\qquad \{u, s\} = \frac{2u'u''' - 3u''^2}{2u'^2} = \tfrac{1}{2}f(s),$$

determined by a function $f(s)$, there belongs a class of congruent curves, lying on the cone $x^2 + y^2 + z^2 = 0$, each curve being determined by a solution u of this equation; the function $f(s)$ is the square of the first curvature of the curve; and the equations of the curve are

$$(47)\qquad x = \frac{1-u^2}{2u'}, \quad y = i\frac{(1+u^2)}{2u'}, \quad z = \frac{u}{u'}.$$

It is readily found that these coordinates are solutions of the equation

$$(48)\qquad \theta''' + \varphi\theta' + \tfrac{1}{2}\varphi'\theta = 0,$$

where the primes indicate differentiation with respect to s.

There is a result analogous to the preceding arising from the consideration of equations (43). Corresponding to (41) and (42) we have

$$(49)\qquad \rho^2 = f^3 f'', \quad s = 2(\psi - ff').$$

Since the second of the expressions (35) vanishes identically, the foregoing constitute a set of characteristic invariants. From (49) we have

$$\frac{ds}{\rho^2} = -2\frac{du}{f^2},$$

*L. c., p. 253.

so that if we introduce a parameter σ defined by

$$\sigma = \tfrac{1}{2}\int\frac{ds}{\rho^2}, \tag{50}$$

and indicate by primes derivatives with respect to σ, we have

$$\{u, \sigma\} = \frac{2u'u''' - 3u''^2}{2u'^2} = 2\rho^2, \tag{51}$$

and equations (43) become

$$\begin{aligned} x &= -\left(1 - \frac{uu''}{2u'^2}\right)^2 u' + \frac{u''^2}{4u'^3}, \\ y &= i\left(1 - \frac{uu''}{2u'^2}\right)^2 u' + \frac{iu''^2}{4u'^3}, \\ z &= \frac{u''}{u'}\left(\frac{uu''}{2u'^2} - 1\right). \end{aligned} \tag{52}$$

Combining these results with the preceding theorem, we obtain the result:

To each solution of a Schwarzian equation

$$\{u, t\} = \tfrac{1}{2}f(t),$$

there correspond two curves on the cone $x^2 + y^2 + z^2 = 0$: *for one of them* t *is the arc and* $f(t)$ *the square of the first curvature; for the other* t *is equal to*

$$\frac{1}{2}\int\frac{ds}{\rho^2}$$

and $f(t)$ *is four times the square of the radius of curvature; when a solution is known the finite equations of the curve can be found without quadrature.*

7. **Curves in General which are not Minimal.**—The foregoing theory of curves on a minimal cone may be applied to any non-minimal curves, since there are associated with any such curve two simple curves on a minimal cone. The coördinates of these curves are

$$\xi = \rho(l + i\lambda), \quad \eta = \rho(m + i\mu), \quad \zeta = \rho(n + i\nu), \tag{53}$$

$$\bar\xi = \rho(l - i\lambda), \quad \bar\eta = \rho(m - i\mu), \quad \bar\zeta = \rho(n - i\nu). \tag{54}$$

Evidently the factor ρ is unessential as far as determining the fact that these curves lie on the cone $x^2 + y^2 + z^2 = 0$. However, this factor enables one to put the expressions in the desired form, when the values for

l, m, n; λ, μ, ν from (32), (33) are substituted. In fact, they reduce by means of (12) to

$$\xi = (1 - u^2)\psi_1', \quad \eta = i(1 + u^2)\psi_1', \quad \zeta = 2u'\psi_1, \tag{55}$$

$$\bar\xi = (1 - \bar u^2)\bar\psi_1', \quad \bar\eta = i(1 + \bar u^2)\bar\psi_1', \quad \bar\zeta = 2\bar u\bar\psi_1', \tag{56}$$

where

$$\psi_1' = \tfrac{1}{2}(u\psi'' - \varphi''), \quad \bar\psi_1' = \tfrac{1}{2}(\bar u\bar\psi'' - \bar\varphi''). \tag{57}$$

From (42) it follows that the arcs s_1 and $\bar s_1$ of the curves (55), (56) are given by

$$s_1 = 2\psi_1 = -(s + c), \quad \bar s_1 = 2\bar\psi_1 = -(\bar s + \bar c). \tag{58}$$

Since these additive constants c, $\bar c$ may be equated to zero, we have the following theorem:

Any analytic curve, which is not minimal, may be represented upon the minimal cone $x^2 + y^2 + z^2 = 0$, in two ways, such that the curvilinear distance of two points on each conical curve is equal to the curvilinear distance between the corresponding points on the given curve. We shall speak of the curves (55), (56) as the *conical representations* of the given curve.

When the expression (57) for ψ_1' is substituted in an equation similar to (41), we obtain the following expression for the curvature of (55)

$$\frac{1}{\rho_1^2} = \frac{1}{\rho^2} - 2\frac{dM}{ds} - M^2, \tag{59}$$

where M denotes the differential invariant

$$M = \frac{i}{\tau} + \frac{1}{\rho}\frac{d\rho}{ds}. \tag{60}$$

From (59) and (46), we have that the determination of curves on the minimal cone which are congruent to the conical representation (55) of a given curve requires the integration of the Schwarzian equation

$$\{u, s\} = \frac{2u'u''' - 3u''^2}{2u'^2} = \frac{1}{2}\left(\frac{1}{\rho^2} - 2\frac{dM}{ds} - M^2\right), \tag{61}$$

where s denotes the arc of the given curve.

In consequence of (57) and (45), the first of equations (58) may be written

$$\varphi'' - u\psi'' = \frac{1}{u'}, \tag{62}$$

where the primes denote differentiation with respect to u and s respect-

ively. Differentiating this equation with respect to s, we obtain

$$(63) \qquad (\varphi'' - u\psi'')' = -\frac{u''}{u'^3},$$

and consequently equation (29) may be written

$$(64) \qquad \psi'' = -\frac{1}{2u'^2}\left(M + \frac{u''}{u'}\right),$$

where primes have the same significance as in (62) and (63).

Conversely, suppose that we have a solution u of equation (61). From (62) and (64) we obtain two functions φ and ψ, which when substituted in (26) and (29) show that the curve determined by these functions has the same intrinsic equations as the given curve. Hence we have the theorem:*

The determination of all curves congruent to a given curve requires the integration of the Schwarzian equation (61) *and quadratures.*

For the other conical representation we have analogous to (61) the equation

$$(65) \qquad \{\bar{u}, s\} = \frac{2\bar{u}'\bar{u}''' - 3\bar{u}''^2}{2\bar{u}'^2} = \frac{1}{2}\left(\frac{1}{\rho^2} + 2\frac{d\bar{M}}{ds} - \bar{M}^2\right),$$

where

$$\bar{M} = \frac{1}{\rho}\frac{d\rho}{d\bar{s}} + \frac{i}{\tau} = -\frac{1}{\rho}\frac{d\rho}{ds} + \frac{i}{\tau}.$$

From (62) and (64) it follows that the solution of (65) corresponding to a solution u of (61) is given by

$$(66) \qquad \bar{u} = u - 2u'^2(Mu' + u'')^{-1}.$$

8. **Congruence of Minimal Curves.** By means of the theory of curves on the minimal cone we are able also to handle the exceptions to the theorems of §4 and to establish a criterion for the congruence of minimal curves.

If we take the equations of a minimal curve in the Weierstrass form†

$$(67) \qquad \begin{aligned} x &= (1 - u^2)f'' + 2uf' - 2f, \\ iy &= -(1 + u^2)f'' + 2uf' - 2f, \\ z &= 2(uf'' - f'), \end{aligned}$$

we obtain on differentiation

$$(68) \qquad \frac{dx}{du} = (1 - u^2)f''', \quad i\frac{dy}{du} = -(1 + u^2)f''', \quad \frac{dz}{du} = 2uf'''.$$

*Cf. Study, l. c., pp. 259, 260.

†L. c.

If we put

(69) $$d\sigma = \sqrt{2f'''}du,$$

it is readily found that

$$\left(\frac{d^2x}{d\sigma^2}\right)^2 + \left(\frac{d^2y}{d\sigma^2}\right)^2 + \left(\frac{d^2z}{d\sigma^2}\right)^2 = 1.$$

Hence σ is the arc of the following curve

(70) $$X = \frac{1-u^2}{\sqrt{2}}\sqrt{f'''}, \quad Y = \frac{i(1+u^2)}{\sqrt{2}}\sqrt{f'''}, \quad Z = \frac{2u}{\sqrt{2}}\sqrt{f'''},$$

which evidently lies on the cone $x^2 + y^2 + z^2 = 0$. In § 6 it was seen that a given curve on this cone admits of only one such representation for given rectangular axes. In like manner it may be shown that a given minimal curve admits of only one such set of equations as (67) for given coördinate axes. Hence the problems of the congruence of minimal curves and of the curves (70) are equivalent.

Comparing equations (70) with (40) we have

$$\psi' = \frac{\sqrt{f'''}}{\sqrt{2}},$$

from which it follows that

$$\frac{2\psi'\psi''' - 3\psi''^2}{(2\,\psi')^4} = \frac{4f'''f^{\mathrm{V}} - 5f^{\mathrm{IV}2}}{32f'''^3}.$$

$$\left(\frac{2\psi'\psi''' - 3\psi''^2}{(2\psi')^4}\right)'\frac{1}{2\psi'} = \frac{\sqrt{2}}{64(f''')^{\frac{9}{2}}}(4f'''^2f^{\mathrm{VI}} - 18f'''f^{\mathrm{IV}}f^{\mathrm{V}} + 15f^{\mathrm{IV}3}).$$

Hence from (44) we have the following theorem:*

The functions

$$J_5 = \frac{4f'''f^{\mathrm{V}} - 5f^{\mathrm{IV}2}}{4f'''^3},$$

$$J_6 = \frac{4f'''^2f^{\mathrm{VI}} - 18f'''f^{\mathrm{IV}}f^{\mathrm{V}} + 15f^{\mathrm{IV}3}}{(f''')^{\frac{9}{2}}}$$

constitute a set of characteristic invariants for a minimal curve.

9. **Algebraic Curves.—Real Curves.** It is evident that if φ and ψ in equation (2) are algebraic functions of u the curve is algebraic. We consider the converse problem.

Suppose that a curve is defined by two algebraic equations

$$f_1(x, y, z) = 0, \quad f_2(x, y, z) = 0.$$

*Lie, l. c., p. 704. Also Vessiot, Sur les courbes minima, Comptes rendus, vol. 140 (1905), pp. 1381–1384. Also Study, l. c., p. 39 and pp. 253–255.

Then y and z are expressible as algebraic functions of x, and consequently u as defined by an equation similar to the first of (8) is an algebraic function of x. In like manner y and z may be shown to be algebraic functions of u. Hence we have the theorem:

The necessary and sufficient condition that a curve defined by equations in the normal form be algebraic is that the coördinates be algebraic functions of the normal parameter.

An immediate consequence of this theorem is that the functions

$$\varphi - u\varphi', \quad \psi', \quad \varphi' - \psi, \qquad \varphi - u\psi$$

must be algebraic functions of u. To these may be added the function $s - 2\psi$, from which follows the theorem:

The necessary and sufficient condition that the arc of an algebraic curve be expressible algebraically in terms of the coördinates of its end points is that φ and ψ be algebraic functions of u.

Algebraic curves of this sort have been called *algebraically rectifiable* by Stäckel.*

Since φ'' and ψ'' are algebraic functions of u for all algebraic curves, and consequently ρ and τ are algebraic, we have the theorem of Salkowski:†

Every algebraic curve which is characterized by an algebraic intrinsic equation $f(s, \rho\ \tau,) = 0$ is algebraically rectifiable.‡

In certain cases it is desirable to know whether or not a curve is real. We shall show that there exists a transformed set of equations in which the functions and parameter are real.

From (2) and (3) it follows that the functions

$$\varphi', \quad u\psi' - \psi, \quad \varphi - u\varphi' + \psi', \quad i(\varphi - u\varphi' - \psi') \tag{71}$$

are real for a real curve, and conversely. If we put

$$u = p + iq,$$

then from (8) it would be possible to determine p and q as functions of the general parameter v, and thus we should obtain a relation between p and q. This is unessential except in that it tells us that we may put

$$\varphi(u) = P'(p) + iQ'(q),$$

*Ueber algebraisch rectifizierbare Raumcurven, Mathematische Annalen, vol. 43 (1893), pp. 171–184. The fundamental theorem of this paper is the following: The necessary and sufficient condition that an algebraic curve be algebraically rectifiable is that it be an evolute of an algebraic curve for which the sine of the total torsion, namely $\int \frac{ds}{\tau}$, is an algebraic function of the coördinates.

†L. c., p. 446.

‡The question of algebraically rectifyable curves has been touched upon also by Darboux, l. c., p. 316.

where the primes indicate differentiation with respect to the argument (the significance of this choice will be seen presently). Since φ' is real, it follows from the equation

$$d\varphi = \varphi'(dp + idq) = P''dp + iQ''dq,$$

that

$$\varphi' = P'' = Q'' = \sigma,$$

where σ is introduced for the sake of subsequent brevity. From these results and the fact that the last two of (71) must be real we find that

$$\psi' = P' - pP'' - i(Q' - qQ'')$$

and finally

$$\psi = 2P - pP' + 2Q - qQ' + i(qP' - pQ').$$

When these values are substituted in (2) and (3) we obtain

$$x = 2(P' - p\sigma), \qquad y = 2(Q' - q\sigma),$$

$$z = \sigma(1 - p^2 - q^2) + 2(pP' - P + qQ' - Q),$$

$$s = \sigma(1 + p^2 + q^2) - 2(pP' - P + qQ' - Q).*$$

In terms of these functions, we have

$$\varphi'' - u\psi'' = \frac{P'''Q'''(1 + p^2 + q^2)}{Q''' + iP'''},$$

$$\frac{1}{\rho^2} = \frac{4}{1 + p^2 + q^2}\left(\frac{1}{P'''^2} + \frac{1}{Q'''^2}\right).$$

$$\bar{u} = \frac{\varphi''}{\psi''} = -\frac{1}{p - iq}.$$

The last of these is an evident consequence of the definition of $\bar{u}$ and u.

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*Cf. Montcheuil, l. c.